Project: Overbooked Flights

In Example 8.6 of the textbook, we investigated the concept of intentionally overbooking flights to guarantee the fullest possible flight. However, in that example, we did not account for the danger the airline risks of having to pay compensation to ticketed passengers who end up "overbooked," i.e., who don't have a seat.

Assuming we are dealing with a Boeing 737-800 in a one-class configuration, we have 189 seats which can be sold to passengers. We will assume that each passenger's decision to make or miss a flight is independent and that each passenger has a 5% chance of missing the flight. We will assume that the airline stands to make $200 for each passenger that actually takes a particular flight.

If the airline sells only 189 tickets, then they can expect approximately 5% of people to miss the flight. This is why airlines will intentionally "overbook" flights by selling more tickets than the number of available seats.

Assuming the airline sells exactly 189 tickets, the probability that 0 passengers arrive for the flight is calculated below.

Coding Corner: Probability of No Showings

```
dbinom(x = 0, size = 189, prob = 0.95)
```

That is, there is nearly a 0% chance that no one shows up for the flight. Under the same assumption, the probability that 1 passenger arrives for the flight is given by the following.

Coding Corner: Probability of One Passenger

```
dbinom(x = 1, size = 189, prob = 0.95)
```

This value is also nearly 0, but it is around a thousand times more likely.

Using the formula for population mean (or Expected Value, $E[X]$) of a discrete random variable in Definition 6.5 (and the following Remark) in Chapter 6, we see that we need not only the probability of each event, but the numeric value of each event.

$$E[X] = \sum_{x \in S} \big(x \cdot P(X = x)\big)$$

If the airline does not charge passengers for missed flights and simply credits an account with their ticket value, then the company will make no revenue if 0 people show up for the flight. However, if 1 person does show up, the company will make $200. If we let X_k be the revenue based on k people making the flight and n be the number of passengers that actually show up for the flight, we see that the expected revenue is given by the following.

$$E[X] = \sum_{k=0}^{189} X_k \cdot P(n = k)$$

This means that the first few terms of $E[X]$ are:

$$0 \cdot P(n = 0) + 200 \cdot P(n = 1)$$

Chapter 8

To get a vector of all values of $P(n = k)$ for $k = 0, 1, 2, \ldots, 189$ we can use:

Coding Corner: Generating Probability Vectors

```
N <- 189
Probk <- dbinom(x = 0:189, size = N, prob = 0.95)
```

where `N` is the number of tickets sold. Moreover, the revenue for $n = k$ is given by $X_k = 200 \cdot k$ and can be set into a vector using the following.

Coding Corner: Generating Revenue Vectors

```
Xk <- 200 * 0:189
```

Bill the Statypus says: To give you an idea of how to do this, try the following code in RStudio.

```
xVec <- c( 1, 2, 3 )
yVec <- c( 0.1, 0.01, 0.001 )
xVec * yVec
sum( xVec * yVec )
```

Use this strategy to tackle the following problems.

a. If the airline sells exactly 189 tickets, find the expected revenue (that is the population mean of the discrete variable as defined in Section 6.1) for the airline using `Xk` and `Probk`. Remember that we need to *multiply* X_k and $P(n = k)$ for each value of k and then *sum* them all up.

b. What is the expected number of passengers that will make the flight and how does this value compare to the value found in part a.?

c. What is the expected revenue if the airline sells 190 tickets if we assume that there is no penalty to the airline for an overbooked passenger other than the $200 that won't be collected from the extra ticket? You should be able to reuse your existing code, but change the value of `N` as needed.

d. What is the probability that all 190 ticketed passengers actually make the flight if we sell 190 tickets?

Bill the Statypus says: More business-minded people may be yelling that we should be discussing profit rather than revenue at this point. However, as we haven't accounted for any operational costs anywhere else in this problem, we will simply "cheat" and treat penalties as negative revenue modifications.

If we have to sell 190 tickets and all passengers show up, we will have to compensate a single passenger. Assume that we compensate this lone unlucky passenger $500, then we can interpret this as a value of $X_{190} = 189 \cdot 200 - 1 \cdot 500$. In this case, if we sell 190 tickets, we get that the expected revenue is modeled by:

$$\sum_{k=0}^{190} X_k \cdot P(n = k) = \sum_{k=0}^{189} X_k \cdot P(n = k) + X_{190} \cdot P(n = 190)$$

However, we must be careful and use 190 for `N` in our calculation for `Probk`:

Coding Corner: Adjusted Probabilities for N

```
N <- 190
Probk <- dbinom(x = 0:189, size = N, prob = 0.95)
```

This gives that $E[X]$ is now found using the following script line:

Coding Corner: Expected Value with Compensation Penalty

```
sum(Xk * Probk) + (189 * 200 - 1 * 500) * dbinom(x = 190, size = 190, prob = 0.95)
```

This value should only be a few cents less than the value found in part c. and greater than the value found in part a. This shows us that even if we have to compensate a passenger who has no seat, selling an extra ticket will increase our expected revenue from the flight.

Sally the Statypus says: Remember, we are imagining this exact scenario playing out over a large number of independent flights and are looking at the long-run average (mean) of the revenue!

e. If we divide the difference of the values in parts a. and c. by the probability in part d., we get the maximum compensation that the airline can give to an overbooked passenger and still expect a long-run monetary gain if they sell 190 tickets. What is this value?

f. Redo the work of part d. to find the expected revenue if airlines expect to pay a federal max benchmark compensation of \$800 per overbooked passenger, changing the penalty profile such that $X_{190} = 189 \cdot 200 - 1 \cdot 800$.

g. If selling 190 tickets increases our expected revenue, selling 191 tickets may increase revenue even more. Carefully recalculating `Probk` with a new value of `N` and adding two terms to `sum(Xk * Probk)` instead of just one, find the expected value of revenue if the airline sells 191 tickets for this flight.

Bill the Statypus says: It may be helpful to use vector operations in R for part g. Consider constructing: `XkOver <- (189 * 200) - (190:N - 189) * 800` to represent the revenue adjustments when $k > 189$, alongside `ProbkOver <- dbinom(x = 190:N, size = N, prob = 0.95)`.

Sally the Statypus says: Don't worry, `dbinom` will handle things correctly even if we pass a boundary value higher than the size parameter. For example, if you calculate `dbinom(190, 189, 0.95)`, it simply returns 0, meaning structural impossibilities won't break your expected value totals!

Reflection: Optimizing the Strategy

h. Continue the pattern laid out in part g. until the expected revenue no longer increases. The last value of `N` previous to a revenue drop is the number of tickets that should maximize revenue. How many tickets should the company sell?

i. Using the cumulative discrete probability techniques we saw in Chapter 8, what is the probability that *any* passenger is left without a seat if the airline sells the exact revenue-maximizing number of tickets found in part h.?

Bill the Statypus says: I know all of the math is correct, but I can't help feeling a bit uneasy knowing we are advocating to sell seats that aren't available. I don't like the idea of a company making money knowing it may be at the cost of upset customers.

Sally the Statypus says: Well, you aren't wrong, but you also have to recognize that if the airlines don't maximize their revenues, they would be forced to charge us much more to fly.

Bill the Statypus says: (grumbling) That doesn't make it right...

Reflection: Personal Feelings

How did this project affect your view on things such as overbooked flights? Is this a simple business decision to optimize the profits of an airline or is it unethical to sell a ticket that may not be honored?